

The Breakdown of Alfvén's Theorem in Ideal Plasma Flows

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Abstract

This paper presents both rigorous results and physical theory on the breakdown of magnetic flux conservation for ideal plasmas, by nonlinear effects. Our analysis is based upon an effective equation for magnetohydrodynamic (MHD) modes at length-scales $> \ell$, with smaller scales eliminated, as in renormalization-group methodology. We prove that flux-conservation can be violated for an arbitrarily small length-scale ℓ , and in the absence of any non-ideality, but only if singular current sheets and vortex sheets both exist and intersect in sets of large enough dimension. This result gives analytical support to and rigorous constraints on theories of fast turbulent reconnection. Mathematically, our theorem is analogous to Onsager's result on energy dissipation anomaly in hydrodynamic turbulence. As a physical phenomenon, the breakdown of magnetic-flux conservation in ideal MHD

is similar to the decay of magnetic flux through a narrow superconducting ring, by phase-slip of quantized flux lines. The effect should be observable both in numerical MHD simulations and in laboratory plasma experiments at moderately high magnetic Reynolds numbers.

Key words: ideal magnetohydrodynamics, line motion, singularities, reconnection

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1 Introduction

In view of the infinite conductivity, every motion (perpendicular to the field) of the liquid in relation to the lines of force is forbidden because it would give infinite eddy currents. Thus the matter of the liquid is “fastened” to the lines of force...

H. ALFVÉN (1942)

It is a fundamental result for an ideal plasma, or perfectly conducting fluid, that magnetic lines of force are “frozen-in” and move with the fluid. This fact was first pointed out by Hannes Alfvén in 1942 [1], in the quote above. About the same time a somewhat stronger result was also observed, that the magnetic flux through a surface moving with a perfectly conducting fluid is conserved. For a good historical review of this early work, see [2]. These properties of magnetic field lines exactly parallel the corresponding properties established by Helmholtz [3] for vortex lines in an ideal, inviscid fluid. The results for ideal magnetohydrodynamics (MHD)—the “frozen-in” property of

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field lines and the conservation of magnetic flux—are often referred to together as “Alfvén’s Theorem”.

One important consequence of these theorems is that magnetic field lines in a perfectly conducting fluid cannot change their topology. In particular, the reconnection of crossed magnetic field lines is forbidden (e.g. see [4]). This poses a bit of a paradox, however. If magnetic field lines are not able to pass through each other more or less freely, then one would expect them to form a complicated tangle that would strongly impede plasma motion, or even thwart it altogether [5]. The theory of ideal plasmas would then be closely analogous to the statistical mechanics of rubber elasticity, where the entanglements of polymer chains imply an infinite set of topological constraints, of which the Gauss linking number is just the simplest [6,7]. However, despite the fact that near-ideal conditions hold in a wide variety of astrophysical situations (interstellar space, the solar corona, etc.), the behavior of these plasmas is not at all “rubber-like” but instead essentially fluid-dynamical. The implication is that reconnection of magnetic field lines occurs at rates that are nearly independent, or even independent, of the value of the resistivity. Explaining this phenomenon of “fast reconnection” is a well-known problem of plasma physics and astrophysics [8,9], with important implications for understanding dynamo action [10], solar flares and coronal mass ejections [11], etc. Since Alfvén’s Theorem prohibits reconnection, any fundamental theory of fast reconnection must explain also the breakdown of those ideal MHD results.

Small Ohmic resistivity (or other non-ideality) implies high magnetic Reynolds numbers, so that laminar solutions of near-ideal MHD equations will be unstable and the plasma motion will generally be turbulent. Theories of fast turbulent reconnection have been proposed [12,13], which predict magnetic-

flux reconnection rates that are completely independent of the resistivity. This implies implicitly the violation of Alfvén’s Theorem under ideal conditions in turbulent plasmas. Such a breakdown of classical conservation laws of the fluid equations under turbulent conditions is not unprecedented. For example, it is well-known that energy is not conserved in the limit of small viscosity for hydrodynamic turbulence, as observed both in laboratory experiments [14,15,16,17] and high-resolution numerical simulations [18,19]. A fundamental explanation for this phenomenon was proposed in 1949 by Lars Onsager [20], who showed that solutions of the ideal incompressible Euler equations do not need to conserve energy if they are sufficiently singular. More precisely, Onsager showed that, if the turbulent velocity field is not differentiable in space but only Hölder continuous with an exponent $\leq 1/3$, then the observed rate of energy dissipation could be explained without any viscosity. See also [21,22,23], and [24] for a recent review. Onsager’s prediction of near-singularities in turbulent velocity fields with Hölder index $\leq 1/3$ has been confirmed by analysis of high-Reynolds number data from experiments and numerical simulations [25,26,27].

Recently, one of us (G.E.) has extended Onsager’s results on inviscid energy dissipation to the breakdown of the Helmholtz-Kelvin Theorem in hydrodynamic turbulence at high Reynolds number [28,29]. The result proved there was that conservation of circulations can break down if the velocity field has near-singularities with Hölder exponent $\leq 1/2$, and it was conjectured on this basis that there would be a “cascade of circulations” for fluid turbulence in three spatial dimensions. This prediction has been confirmed by a high-resolution numerical simulation [30]. The main purpose of the present paper is to establish corresponding mathematical results on the breakdown of

Alfvén’s Theorem for singular solutions of the ideal MHD equations. Despite the similarity of Alfvén’s result to the Helmholtz-Kelvin Theorem in hydrodynamics, there are important differences between the results obtained here and those established in [28,29]. It turns out that it is much harder to violate conservation of magnetic flux than it is to violate conservation of circulation. Hölder continuity exponents that are only moderately small, and far from the most singular behavior in hydrodynamic turbulence, can produce breakdown of the Helmholtz-Kelvin Theorem. The main result proved here can be stated succinctly (but somewhat imprecisely) as follows: *Alfvén’s Theorem can break down in ideal (or near-ideal) magnetohydrodynamics only due to intersecting current sheets and vortex sheets.* The latter are the most singular structures expected to occur in plasma turbulence and only these, acting together, can lead to violation of magnetic flux conservation at a rate which is independent of resistivity. This is a quite striking difference from the hydrodynamic case.

The contents of this paper are as follows: In the following Section 2 we briefly review the formal statement of Alfvén’s Theorem, its derivations, and its implications for magnetic line reconnection. In the next Section 3 we present the “filtering approach”, which is the basis of our entire analysis and explain its relation to theory of distributions (or generalized functions), renormalization group theory, and large-eddy simulation. In Section 4 we prove the main results of the paper. In Section 5 we consider the physical possibilities for breakdown of Alfvén’s Theorem and for ideal reconnection, based upon our rigorous results. In Section 6 we discuss the possibilities for a “cascade of magnetic flux” in MHD turbulence and fast turbulent reconnection. Finally, in Section 7 we restate succinctly our main conclusions.

2 Alfvén's Theorem on Magnetic Flux Conservation

We review here briefly some standard derivations of Alfvén's Theorem(s).

These are based only upon the homogeneous Maxwell equations

$$\nabla \cdot \mathbf{B} = 0, \quad \partial \mathbf{B} / \partial t + \nabla \times \mathbf{E} = 0 \quad (1)$$

and the general Ohm's law

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \mathbf{R}, \quad (2)$$

where \mathbf{R} represents an arbitrary non-ideality and \mathbf{u} is any time-dependent velocity field, not necessarily the solution of any fluid equation. For simplicity, we shall assume here that the velocity is incompressible, $\nabla \cdot \mathbf{u} = 0$, although this is not crucial for the discussion.

The magnetic flux as a Lagrangian variable is defined by

$$\Phi(S, t) = \int_{S(t)} \mathbf{B}(t) \cdot d\mathbf{A} \quad (3)$$

where the initial surface S is smooth and oriented, and $S(t)$ is the surface at later times advected by the velocity field \mathbf{u} . The standard proof of flux conservation uses the easily verified result that

$$(d/dt)\Phi(S, t) = \int_{S(t)} [\partial \mathbf{B} / \partial t - \nabla \times (\mathbf{u} \times \mathbf{B})] \cdot d\mathbf{A}, \quad (4)$$

where the second term in the square bracket represents the change in the flux due to motion of the surface. For example, see [2] or [31], section §38. Taking

the curl of Ohm's law (2) and using Faraday's law (1), gives

$$\partial \mathbf{B} / \partial t = \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times \mathbf{R}. \quad (5)$$

Substituting this result into (4) and using Stokes Theorem gives

$$(d/dt)\Phi(S, t) = - \oint_{C(t)} \mathbf{R} \cdot d\mathbf{x} \quad (6)$$

where $C(t) = \partial S(t)$ is the boundary curve of the advected surface $S(t)$. For $\nabla \times \mathbf{R} = \mathbf{0}$, and in particular for vanishing non-ideality, $\mathbf{R} = \mathbf{0}$, flux conservation immediately follows.

It is a simple consequence of flux-conservation that magnetic flux-tubes—whose surface normal is everywhere perpendicular to the magnetic field—are material surfaces. Since magnetic field lines are intersections of magnetic flux tubes, they must likewise be material lines. Thus, the “frozen-in” property of magnetic field lines is a direct consequence of flux-conservation. More formally, the mathematical condition for line preservation is [2]:

$$[\partial \mathbf{B} / \partial t - \nabla \times (\mathbf{u} \times \mathbf{B})] \times \mathbf{B} = \mathbf{0}.$$

This is the exact analogue of the Helmholtz-Zorawski condition for preservation of vortex lines under fluid advection [32]. Thus, the condition on the non-ideality for “frozen-in” magnetic field lines is $(\nabla \times \mathbf{R}) \times \mathbf{B} = \mathbf{0}$, which is weaker than the condition $\nabla \times \mathbf{R} = \mathbf{0}$ for flux-conservation.

There are other derivations of the Alfvén Theorem that reveal more of its geometric and dynamic significance. The equation (5) with $\nabla \times \mathbf{R} = \mathbf{0}$ is equivalent to the equation

$$\partial_t F + \mathcal{L}_{\mathbf{u}} F = 0$$

where $F = F_{ij} dx_i \wedge dx_j$ is the spatial magnetic 2-form and $\mathcal{L}_{\mathbf{u}}$ is the Lie-derivative for the vector field \mathbf{u} . Flux-conservation then follows immediately from the Lie-derivative Theorem [33]. Another derivation can be based upon the Hamiltonian formulation of ideal magnetohydrodynamics, which possesses an infinite-dimensional gauge symmetry group corresponding to relabelling of fluid particles. In this framework, conservation of magnetic flux for all smooth, oriented surfaces S is a consequence of Noether's theorem for the relabelling symmetry [34].

If the fluid flow is continuous, then the “frozen-in” property forbids any change of magnetic line topology, such as reconnection. A more formal connection with Alfvén's Theorem appears in certain theories of $\mathbf{B} \neq \mathbf{0}$ magnetic reconnection [35,36,37], based upon the *magnetic loop-voltage*

$$V_L = \oint_L \mathbf{R} \cdot d\mathbf{x}, \quad (7)$$

where L is a magnetic field line (which may be closed at infinity). This integral is the same type of voltage which appears in the flux balance (6), breaking flux-conservation, and it vanishes under the same condition $\nabla \times \mathbf{R} = \mathbf{0}$ for which Alfvén's Theorem is valid. The magnetic lines L passing through the region of non-ideality that have extremal values of the loop-voltage are called *reconnection* (or *separator*) *lines*. Under certain assumptions it can be proved that these lines do not undergo reconnection themselves but drive the reconnection of neighboring lines [35,36,37]. Thus, the loop-voltages that produce $\mathbf{B} \neq \mathbf{0}$ reconnection have the same origin as the voltages from any non-ideality that violates Alfvén's Theorem.

3 The Filtering Approach and Large-Scale Flux Balance

The proofs of Alfvén’s Theorem discussed in the preceding section all assume, implicitly, that the solutions \mathbf{u} and \mathbf{b} ¹ of the MHD equations remain smooth in the limit where the resistivity η (or other non-ideality) tends to zero. However, those proofs can break down if the solutions become singular in that limit. To see how this can occur, let us consider the ideal Ohm’s law

$$\mathbf{e} + \mathbf{u} \times \mathbf{b} = \mathbf{0}, \quad (8)$$

and its curl, using Faraday’s law,

$$\partial \mathbf{b} / \partial t = \nabla \times (\mathbf{u} \times \mathbf{b}), \quad (9)$$

in the case that \mathbf{u} and \mathbf{b} are singular. In that case, equation (9), in particular, is not meaningful in a naive sense, because the classical derivative of a non-smooth function is ill-defined. The natural way to interpret equation (9) is in the sense of distributions, which means that it must be multiplied by smooth test functions $\varphi(\mathbf{x})$ and integrated over \mathbf{x} , allowing the curl to be shifted to the test function. (We assume here, for simplicity, that the time-dependence of the solutions is smooth so that test functions $\varphi(\mathbf{x}, t)$ in both variables are not required.) Because equation (9) is quadratically nonlinear, it is not hard to see that it is well-defined, in the sense of distributions, whenever $\mathbf{u}, \mathbf{b} \in L^2$, i.e. are square-integrable, or, physically, have finite energy. In principle, all test functions in a suitable space—e.g. the set $C_c^\infty(\mathbb{R}^3)$ of infinitely-differentiable

¹ Hereafter we use Alfvén velocity variables $\mathbf{b} = \mathbf{B} / \sqrt{4\pi\rho_0}$, with ρ_0 the plasma density, rather than magnetic field variables \mathbf{B} and the corresponding variable $\mathbf{e} = \mathbf{E} / \sqrt{4\pi\rho_0}$ rather than the electric field \mathbf{E} .

functions with compact support—should be considered. Fortunately, it is not necessary to consider *all* of the test functions in the space, but only a subset defined by

$$\varphi_{\ell,\mathbf{x}}(\mathbf{x}') = \ell^{-3} G\left(\frac{\mathbf{x}' - \mathbf{x}}{\ell}\right) \quad (10)$$

for all $0 < \ell < \ell_0$ and $\mathbf{x} \in \mathbb{R}^3$, where G is a particular function that satisfies

$$G \in C_c^\infty(\mathbb{R}^3), \quad G \geq 0, \quad \int d\mathbf{r} G(\mathbf{r}) = 1 \quad (11)$$

One may also substitute here a condition of rapid decay of $G(\mathbf{r})$ at large $|\mathbf{r}|$, faster than any power. For these standard facts of distribution theory, see [38,39], for example.

There is a more physical way to explain this formulation of the equations (8) and (9), which makes clear how Alfvén’s Theorem may be broken. Integrating the solutions \mathbf{u}, \mathbf{b} with respect to the test function (10) yields

$$\bar{\mathbf{u}}_\ell(\mathbf{x}) = \int d\mathbf{r} G_\ell(\mathbf{r}) \mathbf{u}(\mathbf{x} + \mathbf{r}) \quad (12)$$

and similarly for $\bar{\mathbf{b}}_\ell(\mathbf{x})$, where $G_\ell(\mathbf{r}) = \ell^{-3} G(\mathbf{r}/\ell)$. The fields $\bar{\mathbf{u}}_\ell, \bar{\mathbf{b}}_\ell$ are “coarse-grained” at length-scale ℓ or low-pass filtered, retaining information only from scales $> \ell$. These filtered fields satisfy a modified form of Ohm’s law

$$\bar{\mathbf{e}}_\ell + \bar{\mathbf{u}}_\ell \times \bar{\mathbf{b}}_\ell = -\boldsymbol{\varepsilon}_\ell \quad (13)$$

where the *subscale electromotive force (EMF)*

$$\boldsymbol{\varepsilon}_\ell = \overline{(\mathbf{u} \times \mathbf{b})}_\ell - \bar{\mathbf{u}}_\ell \times \bar{\mathbf{b}}_\ell \quad (14)$$

provides an effective non-ideality. The subscale EMF is also sometimes referred to as *turbulent EMF* and, in fact, the present scheme is closely related to the so-called “filtering approach” used in Large-Eddy Simulation (LES) modelling of turbulent flows [40,41]. In this approach the filtered version of equation (9),

$$\partial \bar{\mathbf{b}}_\ell / \partial t = \nabla \times (\bar{\mathbf{u}}_\ell \times \bar{\mathbf{b}}_\ell) + \nabla \times \boldsymbol{\varepsilon}_\ell, \quad (15)$$

would be solved, together with similar filtered equations for the velocity, by employing a closure model for the EMF term $\boldsymbol{\varepsilon}_\ell$.

This same term breaks the validity of flux-conservation for the coarse-grained fields. In fact, the standard derivations of Alfvén’s Theorem now imply that

$$(d/dt) \bar{\Phi}_\ell(S, t) \equiv (d/dt) \int_{\bar{S}_\ell(t)} \bar{\mathbf{b}}_\ell(t) \cdot d\mathbf{A} = \oint_{\bar{C}_\ell(t)} \boldsymbol{\varepsilon}_\ell \cdot d\mathbf{x}. \quad (16)$$

Here our notations are the same as those in the preceding section, except that the surface $\bar{S}_\ell(t)$ and its boundary curve $\bar{C}_\ell(t)$ are defined to be those advected by the filtered velocity $\bar{\mathbf{u}}_\ell$. This “large-scale flux balance” now contains a source/sink term which can violate Alfvén’s Theorem. The physical origin of this phenomenon is an effective “drift velocity” $\Delta \bar{\mathbf{u}}_\ell$ of the field-lines of the large-scale magnetic field $\bar{\mathbf{b}}_\ell$ relative to the plasma velocity $\bar{\mathbf{u}}_\ell$, induced by the subscale EMF. This drift velocity is not uniquely defined, and its component $\Delta \bar{\mathbf{u}}_\ell^\parallel$ parallel to the field lines is largely arbitrary, but it always contains a transverse component [42]:

$$\Delta \bar{\mathbf{u}}_\ell^\perp = \boldsymbol{\varepsilon}_\ell \times \bar{\mathbf{b}}_\ell / |\bar{\mathbf{b}}_\ell|^2. \quad (17)$$

A rather natural prescription to define a drift velocity $\Delta \bar{\mathbf{u}}_\ell$ is proposed in [43]. Because of the additional normal velocity component (17), the field lines

now slip through the plasma and Alfvén’s “frozen-in” property is violated at length-scale ℓ .

The violation of Alfvén’s Theorem so far considered is, in some sense, not “real”. For example, the ideal MHD equations would gain an additional EMF even for a smooth, laminar solution, if that were filtered at any length-scale $\ell > 0$. However, for such solutions the subscale EMF would vanish rapidly in the limit $\ell \rightarrow 0$ (see following section). *We define a real violation of Alfvén’s Theorem for an ideal MHD solution as one which persists in the limit $\ell \rightarrow 0$.* This definition can be justified physically by a Renormalization Group (RG) argument [44]. The dynamical equation (15) for $\bar{\mathbf{b}}_\ell$ is a “renormalized” equation, obtained by integrating out the high-wavenumber modes. In contrast to the “bare” equation (9), it contains only observable quantities. An experiment can measure the velocity and magnetic fields, in fact, only down to a certain spatial resolution corresponding to a length-scale ℓ . If an experimentalist were to attempt to verify magnetic flux conservation, then he would be testing its validity for some coarse-grained fields $\bar{\mathbf{u}}_\ell, \bar{\mathbf{b}}_\ell$ and not for the bare fields \mathbf{u}, \mathbf{b} . Because of the subscale EMF, flux conservation would be violated to some extent for any $\ell > 0$, but the experimentalist would say that flux conservation was verified if it held with increasing accuracy for finer resolutions ℓ . On the contrary, the experimentalist would be forced to say that flux conservation was violated if the effects of the turbulent EMF $\boldsymbol{\varepsilon}_\ell$ did not vanish for arbitrarily small ℓ .

Note that this violation, if present, is entirely an effect of the nonlinearity and not due to any standard non-ideality, such as Ohmic resistivity or other anomalous transport coefficients, such as ambipolar diffusion [45,46], Braginskii viscosity [47,48], etc. The direct effect of such non-ideal terms will be

negligibly small for a sufficiently large length-scale ℓ , much greater than the dissipation length set by resistivity or than internal plasma lengths, such as the ion gyroradius and ion skin depth. For example, in the presence of Ohmic resistivity η , the large-scale effective equation (15) would contain an additional Laplacian term

$$\eta \Delta \bar{\mathbf{b}}_\ell(\mathbf{x}) = \eta \ell^{-2} \int d\mathbf{r} (\Delta G)_\ell(\mathbf{r}) \mathbf{b}(\mathbf{x} + \mathbf{r}).$$

For any solution with finite magnetic energy, $\|\mathbf{b}\|_2 < \infty$, the above equation can be used to provide a bound $\|\eta \Delta \bar{\mathbf{b}}_\ell\|_2 \leq (\text{const.})(\eta/\ell^2)\|\mathbf{b}\|_2$. Therefore, this term vanishes in the limit $\ell \rightarrow \infty$ with η fixed, or $\eta \rightarrow 0$ with ℓ fixed. For a small ratio η/ℓ^2 the Ohmic dissipation term is negligible in the large-scale equation. Of course, for a sufficiently small length-scale ℓ_d , these dissipative, non-ideal effects will be significant. At such small scales, Alfvén's Theorem will be violated due to the dissipative, non-ideal effects. However, at lengths $\ell \gg \ell_d$, Alfvén's Theorem can be violated by the subscale EMF due to the nonlinearity. If both effects exist, then there is no range of length-scales whatsoever where flux-conservation holds.

In the next section we investigate the properties of the solutions \mathbf{u}, \mathbf{b} that permit a persistent effect of the nonlinearity for $\ell > \ell_d$. We find that rather strong singularities are required.

4 Theorems on Flux Conservation

We now prove several simple theorems on sufficient conditions for conservation of magnetic flux, or, equivalently, necessary conditions for the breakdown of flux conservation. A key tool for our analysis is the following formula for the

subscale EMF

$$\boldsymbol{\varepsilon}_\ell(\mathbf{x}) = \int d\mathbf{r} G_\ell(\mathbf{r}) \delta\mathbf{u}(\mathbf{r}; \mathbf{x}) \times \delta\mathbf{b}(\mathbf{r}; \mathbf{x}) - \int d\mathbf{r} G_\ell(\mathbf{r}) \delta\mathbf{u}(\mathbf{r}; \mathbf{x}) \times \int d\mathbf{r} G_\ell(\mathbf{r}) \delta\mathbf{b}(\mathbf{r}; \mathbf{x}) \quad (18)$$

where $\delta\mathbf{u}(\mathbf{r}; \mathbf{x}) = \mathbf{u}(\mathbf{x} + \mathbf{r}) - \mathbf{u}(\mathbf{x})$ is the velocity increment at point \mathbf{x} for separation vector \mathbf{r} and similarly for $\delta\mathbf{b}(\mathbf{r}; \mathbf{x})$. The formula (18) is easily verified by multiplying out the factors and integrating over the separations.

The first point to recognize is that the subscale EMF vanishes nearly everywhere in the limit as $\ell \rightarrow 0$, under very minimal conditions. For example, let us assume just that the total energy (kinetic and magnetic) is finite:

$$E = \frac{1}{2} [\|\mathbf{u}\|_2^2 + \|\mathbf{b}\|_2^2] < \infty, \quad (19)$$

Here $\|\mathbf{u}\|_2^2 = \int_\Lambda d\mathbf{x} |\mathbf{u}(\mathbf{x})|^2$ defines the standard L^2 norm in the flow domain Λ , and similarly for $\|\mathbf{b}\|_2^2$. Then the following result holds:

Proposition 1 *Let $\mathbf{u}, \mathbf{b} \in L^2$. Then $\boldsymbol{\varepsilon}_\ell \xrightarrow{L^1} 0$ as $\ell \rightarrow 0$.*

If the flow domain Λ is infinite, it is more natural to substitute the conditions that \mathbf{u}, \mathbf{b} have locally finite energy densities and then the convergence of $\boldsymbol{\varepsilon}_\ell$ to zero is in the local L^1 sense.

Proof of Proposition 1: A standard density argument from real analysis shows that the L^2 -norms of the increments, $\|\delta\mathbf{u}(\mathbf{r})\|_2$ and $\|\delta\mathbf{b}(\mathbf{r})\|_2$, are continuous in \mathbf{r} and, in particular, vanish as $\mathbf{r} \rightarrow \mathbf{0}$. In fact, by the reverse triangle inequality,

$$|\|\delta\mathbf{u}(\mathbf{r})\|_2 - \|\delta\mathbf{u}(\mathbf{r}')\|_2| \leq \|\delta\mathbf{u}(\mathbf{r}) - \delta\mathbf{u}(\mathbf{r}')\|_2 = \|\mathbf{u}(\cdot + \mathbf{r}) - \mathbf{u}(\cdot + \mathbf{r}')\|_2.$$

Since smooth functions are dense in L^2 , there exists a smooth function \mathbf{u}^* so that $\|\mathbf{u} - \mathbf{u}^*\|_2 < \epsilon$ for any $\epsilon > 0$. Thus, by triangle inequality,

$$\|\mathbf{u}(\cdot + \mathbf{r}) - \mathbf{u}(\cdot + \mathbf{r}')\|_2 \leq 2\epsilon + \|\mathbf{u}^*(\cdot + \mathbf{r}) - \mathbf{u}^*(\cdot + \mathbf{r}')\|_2.$$

Since \mathbf{u}^* is smooth, we get

$$\limsup_{\mathbf{r}' \rightarrow \mathbf{r}} \|\mathbf{u}(\cdot + \mathbf{r}) - \mathbf{u}(\cdot + \mathbf{r}')\|_2 \leq 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, it follows that $\lim_{\mathbf{r}' \rightarrow \mathbf{r}} \|\delta\mathbf{u}(\mathbf{r}')\|_2 = \|\delta\mathbf{u}(\mathbf{r})\|_2$, completing the argument. To finish the proof, we observe by the triangle and Hölder inequalities applied to (18) that

$$\|\boldsymbol{\varepsilon}_\ell\|_1 \leq \int d\mathbf{r} G_\ell(\mathbf{r}) \|\delta\mathbf{u}(\mathbf{r})\|_2 \|\delta\mathbf{b}(\mathbf{r})\|_2 + \int d\mathbf{r} G_\ell(\mathbf{r}) \|\delta\mathbf{u}(\mathbf{r})\|_2 \int d\mathbf{r} G_\ell(\mathbf{r}) \|\delta\mathbf{b}(\mathbf{r})\|_2.$$

Since $\|\delta\mathbf{u}(\mathbf{r})\|_2$ is continuous and vanishes at $\mathbf{r} = \mathbf{0}$, it follows that $\lim_{\ell \rightarrow 0} \|\boldsymbol{\varepsilon}_\ell\|_1 = 0$. QED

This result is in sharp contrast to that for the analogous “vortex force” in the hydrodynamic case, $\mathbf{f}_\ell \equiv \overline{(\mathbf{u} \times \boldsymbol{\omega})}_\ell - \overline{\mathbf{u}}_\ell \times \overline{\boldsymbol{\omega}}_\ell$, which does not need to vanish in the limit as $\ell \rightarrow 0$, even if the velocity \mathbf{u} is continuous. It was proved in [28] that \mathbf{f}_ℓ only needs to vanish if \mathbf{u} is Hölder continuous of order greater than $1/2$. The Proposition 1 tells us that the limit of the EMF along a subsequence for $\ell \rightarrow 0$ vanishes except on a set of Lebesgue measure zero, when the total energy is finite. However, this result does not imply conservation of flux for every curve C , since such sets have (three-dimensional) Lebesgue measure zero and the line-integral of $\boldsymbol{\varepsilon}_\ell$ on certain choices of the loop C might not vanish in the limit.

To get a result on flux conservation, let us prove a spatially local version of the previous proposition, under stronger assumptions:

Proposition 2 *If either \mathbf{u} or \mathbf{b} is continuous at point \mathbf{x} and if the other is bounded, then $\boldsymbol{\varepsilon}_\ell(\mathbf{x}) \rightarrow 0$ as $\ell \rightarrow 0$.*

Proof of Proposition 2: Without loss of generality, let us assume that \mathbf{u} is continuous at \mathbf{x} and that $|\mathbf{b}(\mathbf{x}')| \leq B$ for $\mathbf{x}' \in \Lambda$. Then, $|\delta\mathbf{b}(\mathbf{r}; \mathbf{x})| \leq 2B$ and by the triangle inequality

$$\begin{aligned} |\boldsymbol{\varepsilon}_\ell(\mathbf{x})| &\leq \int d\mathbf{r} G_\ell(\mathbf{r}) |\delta\mathbf{u}(\mathbf{r}; \mathbf{x})| |\delta\mathbf{b}(\mathbf{r}; \mathbf{x})| + \int d\mathbf{r} G_\ell(\mathbf{r}) |\delta\mathbf{u}(\mathbf{r}; \mathbf{x})| \int d\mathbf{r} G_\ell(\mathbf{r}) |\delta\mathbf{b}(\mathbf{r}; \mathbf{x})| \\ &\leq 4B \int d\mathbf{r} G_\ell(\mathbf{r}) |\delta\mathbf{u}(\mathbf{r}; \mathbf{x})|. \end{aligned}$$

It follows then that $\lim_{\ell \rightarrow 0} \boldsymbol{\varepsilon}_\ell(\mathbf{x}) = \mathbf{0}$, by continuity of \mathbf{u} at \mathbf{x} . Note that, if the filter kernel G is compactly supported in space, then we need only assume that \mathbf{b} is bounded in a small neighborhood of \mathbf{x} . QED

This result has the very important implication that, in order to get a non-vanishing EMF in the limit $\ell \rightarrow 0$ at a point \mathbf{x} , *both* the velocity *and* the magnetic field must be irregular there, at least discontinuous or even unbounded.

We can now deduce the following simple consequence for flux conservation:

Corollary 1 *Let C be a closed, oriented, and rectifiable curve, and let \mathbf{u}, \mathbf{b} be bounded functions in a neighborhood of C , such that at least one of them is continuous at every point of C except for a set of one-dimensional Hausdorff measure H^1 equal to zero. Then,*

$$\lim_{\ell \rightarrow 0} \oint_C \boldsymbol{\varepsilon}_\ell \cdot d\mathbf{x} = 0.$$

An equivalent statement of this result is that, for Alfvén's Theorem to break down, one of the following conditions must hold: either (i) the curve C must be non-rectifiable, or (ii) at least one of \mathbf{u} or \mathbf{b} must be unbounded on C , or

else (iii) the curve C and the set of discontinuities $\mathcal{D} = \mathcal{D}_{\mathbf{u}} \cap \mathcal{D}_{\mathbf{b}}$ of both \mathbf{u} and \mathbf{b} must intersect in a set of finite length. More technically, it is required that

$$H^1(C \cap \mathcal{D}) > 0, \quad (20)$$

where H^1 is the one-dimensional Hausdorff measure on subsets of \mathbb{R}^3 .

We shall return later to conditions (i) and (ii) in our discussion of turbulent MHD flows. Here we focus on condition (iii), showing by an example that it can indeed lead to violation of flux-conservation:

Example 1 *Let*

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \frac{1}{2} \Delta u_0 \operatorname{sign}(y) \hat{\mathbf{i}} \\ \mathbf{b}(\mathbf{x}) &= \frac{1}{2} \Delta b_0 \operatorname{sign}(x \cos \varphi + y \sin \varphi) [\hat{\mathbf{j}} \cos \varphi - \hat{\mathbf{i}} \sin \varphi] \end{aligned}$$

and let G be any spherically symmetric (or even cylindrically symmetric) filter kernel. Then, on the z -axis

$$\varepsilon_\ell(0, 0, z) = \frac{\Delta u_0 \Delta b_0}{2\pi} \sigma(\varphi) \cos(\varphi) \hat{\mathbf{k}}, \quad (21)$$

independent of ℓ , while at all other points $\lim_{\ell \rightarrow 0} \varepsilon_\ell(\mathbf{x}) = 0$. Here σ is the 2π -periodic function defined by

$$\sigma(\varphi) = \begin{cases} \varphi & -\pi/2 < \varphi < \pi/2 \\ \pi - \varphi & \pi/2 < \varphi < 3\pi/2 \end{cases}$$

This example consists of a vortex sheet and a current sheet intersecting in a line, with an angle of $\pi/2 - \varphi$ between them. The vortex sheet has strength Δu_0 and lies in the xz -plane, while the current sheet has strength Δb_0 and lies in a plane obtained by rotating the yz -plane around the z -axis by angle φ .

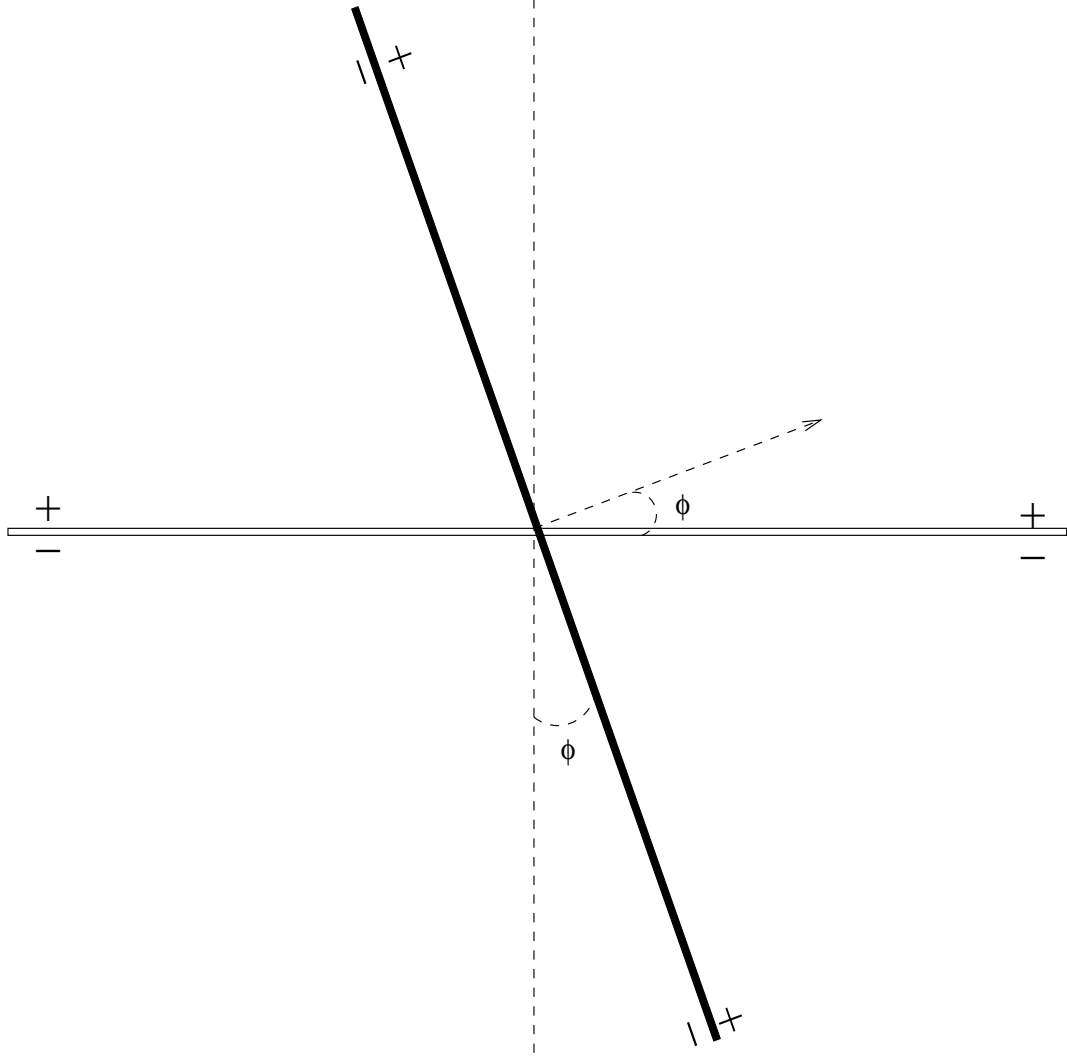


Fig. 1. An intersecting vortex sheet and current sheet viewed along the axis of intersection (the z -axis). The white strip represents the vortex sheet in the xz -plane. The black strip represents the current sheet, in the plane obtained by rotating the yz -plane by angle φ around the z -axis. The \pm labels on the sides of the sheets indicate the values of the sign functions in the definition of the velocity and magnetic field for this example.

(See Figure 1.) To establish (21), note first that for any cylindrically symmetric filter G the filtered functions $\bar{\mathbf{u}}_\ell(\mathbf{x})$, $\bar{\mathbf{b}}_\ell(\mathbf{x})$ vanish on the z -axis. (In fact, they vanish on the entire sheets of discontinuity of \mathbf{u} and \mathbf{b} , respectively.) Thus,

$\boldsymbol{\varepsilon}_\ell = \overline{(\mathbf{u} \times \mathbf{b})}_\ell$ on the z -axis. Furthermore, in cylindrical coordinates (r, θ, z) ,

$$\mathbf{u}(\mathbf{x}) \times \mathbf{b}(\mathbf{x}) = \frac{1}{4} \Delta u_0 \Delta b_0 \operatorname{sign}(\sin(\theta)) \operatorname{sign}(\cos(\theta - \varphi)) \cos(\varphi) \hat{\mathbf{k}}.$$

Thus, for any cylindrically-symmetric filter (not depending upon θ),

$$\boldsymbol{\varepsilon}_\ell(0, 0, z) = \frac{1}{8\pi} \Delta u_0 \Delta b_0 \cos(\varphi) \hat{\mathbf{k}} \int_0^{2\pi} d\theta \operatorname{sign}(\sin(\theta)) \operatorname{sign}(\cos(\theta - \varphi)).$$

The integral has the value $4\sigma(\varphi)$, giving the result (21). Note that $\lim_{\ell \rightarrow 0} \boldsymbol{\varepsilon}_\ell(\mathbf{x}) = 0$ off the z -axis by our Proposition 2. In fact, this is easy to see directly using the constancy of \mathbf{u} and \mathbf{b} off the sheets, so that $\delta \mathbf{u} = \delta \mathbf{b} = \mathbf{0}$, and formula (18).

In the above example, consider a smooth loop C that has one segment consisting of an interval along the z -axis of length L_z . Then for any such loop C ,

$$\lim_{\ell \rightarrow 0} \oint_C \boldsymbol{\varepsilon}_\ell \cdot d\mathbf{x} = \frac{\Delta u_0 \Delta b_0}{2\pi} L_z \sigma(\varphi) \cos(\varphi),$$

if the orientation of the curve is upward along the segment on the z -axis. The function $\sigma(\varphi)$ resembles the trigometric sine function, but is piecewise linear, and its product with $\cos(\varphi)$ is nonzero except for φ an integer multiple of $\pi/2$. The maximum value in the first quadrant, $0 < \varphi < \pi/2$, occurs for the solution of $\cot \varphi_* = \varphi_*$, or $\varphi_* \doteq 0.8603$, an angle a bit larger than $\pi/4$. In any case, for any angle φ not an integer multiple of $\pi/2$ and for an infinite set of loops, flux through the loop C is not conserved, instantaneously, in the limit as $\ell \rightarrow 0$.

5 Physical Breakdown of Alfvén's Theorem?

In our discussion thus far we have used only the homogeneous Maxwell equations (1) and the ideal Ohm's law (8) and, in particular, their consequence, equation (9), which we repeat here, for convenience, in a somewhat different form:

$$\partial \mathbf{b} / \partial t + (\mathbf{u} \cdot \nabla) \mathbf{b} = (\mathbf{b} \cdot \nabla) \mathbf{u}. \quad (22)$$

To fully describe an ideal plasma, there must be adjoined also the momentum equation:

$$\partial \mathbf{u} / \partial t + (\mathbf{u} \cdot \nabla) \mathbf{u} = (\mathbf{b} \cdot \nabla) \mathbf{b} - \nabla \tilde{p} \quad (23)$$

where $\tilde{p} = p + b^2/2$ combines the hydrodynamic and magnetic pressure. We now examine the possibilities for the breakdown of Alfvén's Theorem in the context of (22) and (23), the equations of ideal magnetohydrodynamics.

Our Proposition 2 shows that discontinuous solutions are necessary for the breakdown of flux conservation (or, even worse, unbounded solutions). Of course, it is well-known that the ideal MHD equations possess solutions that are piecewise smooth with jump discontinuities on a smooth surface \mathcal{D} . A compressible plasma possesses a richer set of such solutions (including shocks), but here we restrict ourselves to incompressible fluids. In that case the jump conditions at the surface \mathcal{D} of discontinuity are [49,50]:

$$\Delta u_n = \Delta b_n = 0 \quad (24)$$

$$b_n \Delta \mathbf{u}_t = (u_n - v_n) \Delta \mathbf{b}_t \quad (25)$$

$$(u_n - v_n)\Delta\mathbf{u}_t = b_n\Delta\mathbf{b}_t, \quad \Delta\tilde{p} = 0 \quad (26)$$

Here Δf for any quantity f denotes its jump across \mathcal{D} . u_n, b_n are the components of \mathbf{u}, \mathbf{b} locally normal to the surface and they must have no discontinuity, by the divergence-free condition. The discontinuities in the tangential components $\mathbf{u}_t, \mathbf{b}_t$ are related by (25), (26), which follow from (22),(23). Here v_n denotes the velocity of the surface \mathcal{D} normal to itself. These equations imply that $|u_n - v_n| = |b_n|$ and thus allow two classes of solutions [49,50]. The first class has non-zero mass flow across the surface \mathcal{D} of discontinuity:

$$|u_n - v_n| = |b_n| \neq 0 \implies \Delta\mathbf{u}_t = \pm\Delta\mathbf{b}_t. \quad (27)$$

The second type has no mass flow across \mathcal{D} :

$$|u_n - v_n| = |b_n| = 0 \implies \Delta\mathbf{u}_t, \Delta\mathbf{b}_t \text{ arbitrary} \quad (28)$$

The second type is called generally a “tangential discontinuity”, or, if both $\Delta\mathbf{u}_t$ and $\Delta\mathbf{b}_t$ are non-zero, a *current-vortex sheet*.

It is interesting to observe that a single such structure, in isolation, can lead to no breakdown of Alfvén’s Theorem. If the filter kernel G is spherically symmetric, then it is easy to see that at a point $\mathbf{x} \in \mathcal{D}$,

$$\lim_{\ell \rightarrow 0} \bar{\mathbf{u}}_\ell(\mathbf{x}) = \frac{1}{2}[\mathbf{u}_+(\mathbf{x}) + \mathbf{u}_-(\mathbf{x})],$$

where $\mathbf{u}_+(\mathbf{x}), \mathbf{u}_-(\mathbf{x})$ are the values of velocity \mathbf{u} approaching \mathbf{x} from either side of \mathcal{D} . Using the similar results for $\bar{\mathbf{b}}_\ell, \overline{(\mathbf{u} \times \mathbf{b})}_\ell$ gives immediately that

$$\lim_{\ell \rightarrow 0} \boldsymbol{\varepsilon}_\ell(\mathbf{x}) = \frac{1}{4}\Delta\mathbf{u}_t(\mathbf{x}) \times \Delta\mathbf{b}_t(\mathbf{x}) \quad (29)$$

for $\mathbf{x} \in \mathcal{D}$. Of course, the limit is zero for $\mathbf{x} \notin \mathcal{D}$. For the type of discontinuity with mass-flow, condition (27) implies that the limit is zero also on \mathcal{D} . The second type of discontinuity, however, a general current-vortex sheet, can have a nonzero limit for $\boldsymbol{\varepsilon}_\ell$ over the entire two-dimensional surface \mathcal{D} . Nevertheless, this EMF cannot lead to a violation of Alfvén’s Theorem in the limit, because it is everywhere normal to the surface and no non-vanishing line-integral is possible. Thus, our Example 1 of the preceding section, with a pair of intersecting sheets, seems the simplest possibility for breakdown of flux conservation.

Both current sheets and vortex sheets are commonly observed in numerical simulations of near-ideal MHD equations, for both two-dimensions (2D) and three-dimensions (3D). We know of no evidence from these simulations for any worse singularity with unbounded $|\mathbf{u}|$ or $|\mathbf{b}|$. This seems to indicate that the condition (ii) of our Corollary 1 is only an academic possibility, not physically realized. We shall review here briefly some of the available numerical results, with no attempt at completeness.

Current sheets and vortex sheets have been observed to develop in a variety of simulations of freely-decaying 2D-MHD, in approximately ideal conditions. Initial conditions that have been employed include the Orszag-Tang vortex or slight modifications [51,52,53] and random initial conditions [52]. The current sheets and vortex sheets appear usually in very close proximity. A rather successful analytical theory was developed in [54] for the formation of these structures near X -type magnetic null points. Note, however, that there can be no inviscid breakdown of Alfvén’s Theorem in 2D, from current sheets and vortex sheets, or from any other type of singularity. Indeed, by its definition, the subscale EMF vector $\boldsymbol{\varepsilon}_\ell$ points always normal to the 2D plane and

therefore no non-vanishing line integral is possible. This is in contrast to the hydrodynamic case, where a breakdown of the Kelvin Theorem is possible in 2D [28,29,30].

Current sheets and vortex sheets are also observed to develop in simulations of 3D-MHD, at moderate magnetic Reynolds numbers. These have been observed in freely-evolving flows initialized by a 3D extension of the Orszag-Tang vortex [55,56], by linked flux rings [56], or by random initial conditions [55], and also in forced 3D MHD turbulence [57]. The latter work [57] did not study \mathbf{u} , \mathbf{b} fields but instead Elsasser variables $\mathbf{z}^\pm = \mathbf{u} \pm \mathbf{b}$, further decomposed into contributions from shear-Alfvén and pseudo-Alfvén modes. It was found that all four of these fields form strong sheet-like singularities in close proximity. Of course, the subscale EMF may be written also in terms of Elsasser fields, as

$$\boldsymbol{\varepsilon}_\ell = \frac{1}{2} \left[\overline{(\mathbf{z}^- \times \mathbf{z}^+)}_\ell - \bar{\mathbf{z}}_\ell^- \times \bar{\mathbf{z}}_\ell^+ \right], \quad (30)$$

and intersecting sheets of discontinuity of \mathbf{z}^+ and \mathbf{z}^- can generate a non-vanishing EMF. The simulations in [55] found closely associated sheets for \mathbf{u} , \mathbf{b} and they concluded that “vorticity and current display similar features and are usually intense in adjacent regions.” This paper also studied the dynamics of these structures, their formation and persistence. The latter is an important issue, since intersection of vortex and current sheets are required, not only at an instant but also over some interval of time. By equation (16), the large-scale magnetic fluxes at two subsequent times $t' > t$ are related by

$$\bar{\Phi}_\ell(S, t') - \bar{\Phi}_\ell(S, t) = \int_t^{t'} d\tau \oint_{\bar{C}_\ell(\tau)} \boldsymbol{\varepsilon}_\ell(\tau) \cdot d\mathbf{x}. \quad (31)$$

(if all non-ideal effects may be ignored at scale ℓ .) Thus, the magnetic flux will be conserved if the line-integral of the EMF is non-zero only for a set of times of measure zero. Our Example 1 in the previous section does not address this issue, because the configuration of intersecting vortex and current sheets employed there is not a stationary solution of the ideal MHD equations.

Nevertheless, we conjecture that magnetic flux conservation may indeed be broken in ideal MHD by nonlinear effects. As a physical phenomenon, it should be analogous to the decay of magnetic flux trapped in a narrow superconducting ring. According to both theory [58,59,60] and experiment [61,62] this decay is due to nucleation of quantized magnetic flux lines (by thermal, quantum, or other fluctuations), which locally destroy the superconducting state. The quantized flux lines migrate out of the ring, allowing the relative phase across the point of escape to slip by 2π and generating a voltage pulse around the ring. Here it is important that the quantized flux lines need not move with the local superfluid velocity, due primarily to drag forces generated by their interaction with the background excitations (quasi-particles and holes) [63]. The physics in ideal MHD is similar, with the necessary singularities provided by the (intersections of) current sheets and vortex sheets. In the presence of such singularities, the large-scale magnetic field lines do not move with the plasma velocity at the same scale but gain a “slip velocity” due to their interaction with the subscale modes: see eq. (17). The diffusion of the lines of force of the large-scale magnetic field out of the advected loop implies a voltage pulse around the loop, which can lead to violation of flux conservation. There is not only a physical similarity of this process with quantum-phase slip in superconductors but also, as we discussed in [29], a fairly close formal analogy as well.

To complete this section we would like to make a few comments on the relation of our results to various theories of magnetic line reconnection in MHD. Our simple Example 1 provides conditions similar to those required in several such theories. In quasi-2D reconnection, there is an X -type magnetic neutral line along which the parallel component of the electric field is non-vanishing [4]. Such neutral lines are not structurally stable in 3D, so theories of 3D reconnection are often based instead upon neutral points (magnetic nulls) which are stable [64,65]. These theories require a non-vanishing line-integral of the electric field along magnetic field lines that connect pairs of nulls (null-null lines). Finally, theories of $\mathbf{B} \neq \mathbf{0}$ reconnection [35,36,37] are based upon magnetic field-lines of maximum loop voltage (eq.(7)). In our Example 1, the current sheet is a neutral sheet for the large-scale magnetic field $\bar{\mathbf{b}}_\ell$, for any $\ell > 0$. In particular, a neutral line exists along which there is a non-vanishing integral of the electric field. Adding a smooth external magnetic field with only a z -component, and non-vanishing on the z -axis, makes this null line a magnetic field line with maximal loop-voltage in the limit $\ell \rightarrow 0$. By adding an appropriate smooth external magnetic field, this line can also be converted to a null-null field line. However, all of these constructions are contrived and clearly inadequate as a general model of fast reconnection (i.e. with rates independent of resistivity). In the next section we shall indicate what we believe are some missing ingredients of such a theory.

The most important implication for theories of fast reconnection follows from Proposition 2 and its Corollary 1. These show that vortex sheets are equally important as current sheets to obtain non-vanishing reconnection in the ideal limit. Any successful theory must involve essentially the coincident singularities of the velocity field \mathbf{u} and magnetic field \mathbf{b} .

6 Turbulent Cascade of Magnetic Flux

The results in the present paper are not specific to turbulent plasma flows but this is one of their most interesting areas of application. The necessary ingredients to violate Alfvén’s Theorem—singular current and vortex sheets—seem to exist in MHD turbulence in the limit of high fluid and magnetic Reynolds numbers. Therefore, we expect that magnetic-flux conservation breaks down under turbulent conditions. Analogous to the case of fluid velocity circulations discussed in [28,29,30], we may term this a “cascade of magnetic-flux”. However, the term “cascade” is not as well warranted here, since the scale-locality of the process is in serious doubt. Following the discussion in [29,66], the turbulent EMF $\boldsymbol{\varepsilon}_\ell(\mathbf{x})$ is infrared (IR) local-in-scale if the Hölder exponents of \mathbf{u}, \mathbf{b} at the point \mathbf{x} satisfy $h_u < 1$ and $h_b < 1$. Similarly, the turbulent EMF is ultra-violet (UV) local-in-scale at the point \mathbf{x} if the Hölder exponents there satisfy $h_u > 0$ and $h_b > 0$. This means that the EMF is UV-local away from discontinuities. However it is precisely due to these points that the flux-conservation is violated! It is possible that UV-locality still holds at such points, but it requires extensive cancellations between the contributions from length-scales $\ll \ell$. This is unlikely if the current sheets and vortex sheets are highly coherent structures at all length-scales. Thus, it is more likely that UV locality is only marginal there. This complicates the task of developing adequate theoretical models for the turbulent EMF $\boldsymbol{\varepsilon}_\ell(\mathbf{x})$. For example, the “multi-scale gradient” expansion that was developed in [67] and applied to the circulation cascade in [29] is based upon UV-locality. Its application to the cascade of magnetic-flux in MHD turbulence may thus be only qualitatively successful for $\ell \rightarrow 0$.

Another complication in turbulent MHD flows is that material curves $C(t)$

advected by a velocity field which is not differentiable but only Hölder continuous are expected to become fractal, with a Hausdorff dimension > 1 [68,69]. This is likely to occur in MHD turbulence in the limit of infinite magnetic and fluid Reynolds numbers. Fractality of material curves provides another potential mechanism for breakdown of Alfvén's Theorem, since fractal curves are non-rectifiable (condition (i) in Corollary 1.) In fact, for fractal curves and surfaces it is not even clear how to *define* integrals such as the magnetic-flux (3) or the loop-voltage (6). One possibility is to write, for example,

$$\oint_{C(t)} \mathbf{R}(\mathbf{x}, t) \cdot d\mathbf{x} = \oint_C \mathbf{R}_L(\mathbf{a}, t) \cdot d\mathbf{x}(\mathbf{a}, t) \quad (32)$$

where $\mathbf{x}(\mathbf{a}, t)$ is the Lagrangian flow map, defined by

$$(d/dt)\mathbf{x}(\mathbf{a}, t) = \mathbf{u}(\mathbf{x}(\mathbf{a}, t), t), \quad \mathbf{x}(\mathbf{a}, t_0) = \mathbf{a},$$

$\mathbf{R}_L(\mathbf{a}, t) = \mathbf{R}(\mathbf{x}(\mathbf{a}, t), t)$ is the non-ideality in a Lagrangian frame, and the integral on the right side of (32) is defined over the initial loop C at time t_0 . It was shown by Young [70,71] that this integral exists in the Stieltjes-sense if the minimal Hölder exponents h_R of $\mathbf{R}_L(\mathbf{a}, t)$ and h_x of $\mathbf{x}(\mathbf{a}, t)$ satisfy $h_R + h_x > 1$. We expect that condition (i) can be removed in Corollary 1 by an application of such ideas. An interesting laboratory in which to study this question is the Kazantsev-Kraichnan dynamo model [72,73], for the case of non-smooth advecting velocity field [74,75,76]. It is expected that advected curves and surfaces in this model will become fractal, just as in real turbulence [28]. However, the advecting random velocity field is Gaussian and monofractal, so that there are no vortex sheets (and perhaps no current sheets). Thus, the effect of fractality of material objects can be studied in isolation. We conjecture that the magnetic flux is strictly conserved for *all* surfaces in the

Kazantsev-Kraichnan model, when the advecting velocity is non-smooth but Hölder continuous.

Although we expect no direct effect of fractality of the surface $S(t)$ (or of its boundary $C(t)$) on conservation of magnetic flux, there can be an indirect effect. If the Hausdorff dimension of $C(t)$ is > 1 , then it increases the probability of a nontrivial intersection of the loop with the discontinuity set \mathcal{D} of \mathbf{u} and \mathbf{b} (cf. eq.(20)). If the Hausdorff dimensions of the current-vortex sheets are > 2 , then the dimension of their typical intersection \mathcal{D} will be > 1 and this will also enhance the probability of condition (20) being satisfied. Most phenomenological models of intermittency in MHD turbulence have assumed that the Hausdorff dimension of the sheets is exactly equal to 2 [77,78,79,80]. However, it is plausible to expect that turbulent advection on all scales will lead to wrinkling of the sheets, increasing their dimensionality to values > 2 .

A similar effect will appear due to the *spontaneous stochasticity* of magnetic field-lines in the limit of infinite magnetic Reynolds numbers. Field lines $\boldsymbol{\xi}(\sigma, t)$ are defined in principle at each time t by solving the ODE (in the parameter σ related to arclength s by $ds = |\mathbf{b}|d\sigma$)

$$(d/d\sigma)\boldsymbol{\xi}(\sigma, t) = \mathbf{b}(\boldsymbol{\xi}(\sigma, t), t), \quad \boldsymbol{\xi}(0, t) = \boldsymbol{\xi}_0 \quad (33)$$

for the given magnetic field $\mathbf{b}(\mathbf{x}, t)$. However, in the limit of infinite magnetic Reynolds number, the magnetic field is non-smooth and, in fact, probably nowhere-differentiable. In that case, the solutions of (33) are not only fractal but also presumably random. This line-stochasticity can arise mathematically from the non-uniqueness of the solutions of the initial-value problem (33) when \mathbf{b} is non-smooth. Physically, it corresponds to a turbulent diffusion in the ar-

length parameter s , analogous to Richardson diffusion of material particles in hydrodynamic turbulence [81]. This phenomenon of “spontaneous stochasticity” was first noted for Lagrangian trajectories in the Kraichnan model of random advection [82,83]. It has since been rigorously proved in the Kraichnan model that the solutions for the Lagrangian trajectories correspond to a random process, with a *fixed* initial condition \mathbf{x}_0 for the fluid particle and a *fixed* advecting velocity \mathbf{u} [84,85]. These considerations carry over plausibly also to the equation (33) for the magnetic field-lines. Such stochastic effects will increase the likelihood of magnetic field-lines intersecting the singular set \mathcal{D} . Note that this type of random field-line wandering is a crucial part of current theories of fast turbulent reconnection [12,13].

7 Conclusions

The results presented in this paper support theories of fast turbulent reconnection [12,13,5], in a general way, but also place rigorous constraints upon them. A basic assumption of those theories is that Alfvén’s Theorem may be violated in the limit of vanishing resistivity. We have shown that this is possible by an analysis of the MHD equations for an ideal plasma. However, in contrast to Kelvin’s circulation theorem, which is rather easily violated [28,29,30], Alfvén’s theorem on magnetic-flux conservation is much more robust. We have proved that violations of it are only possible, essentially, if singular vortex sheets and current sheets have intersections with high enough dimension and persist long enough in time. These results should help to guide further theoretical, numerical, and experimental work.

We have shown, in particular, that it is crucial to understand the physical

properties of the subscale EMF, defined by eq.(14). At large enough length-scales ℓ , this quantity is the critical driver of magnetic line reconnection and any non-ideality at small scales is irrelevant. To explain the fast rates of reconnection observed in astrophysical situations, where many decades of inertial range are often observed, a quantitative theory for the turbulent EMF must be developed.

The predicted breakdown of magnetic-flux conservation in ideal MHD, as a physical phenomenon, is closely analogous to the decay by quantum phase-slip of magnetic flux confined in a superconducting ring. It should be observable both in numerical simulations and in laboratory experiments at moderately high Reynolds numbers.

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